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RESEARCH MEMORANDUM

APPLICATION OF STATISTICAL THEORY TO BEAM-RIDER

GUIDANCE IN THE PRESENCE OF NOISE.

I - WIENER FILTER THEORY

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NATIONAL ADVISORY COMMITTEE
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SUMMARY

A study has been made of the application of Wiener filter theory to the design of a beam-rider guidance system operating in the presence of glint noise. Target and missile motions are restricted to the same plane. The Wiener theory is then used to establish the theoretical lower limit of root-mean-square error and the corresponding desired transfer-function characteristics. It is shown that although the practical achievement of these results is restricted by saturation effects, the theory is useful, with suitable modifications, as a guide in system design. Such modifications have been applied to the design of systems for which the optimum filtering is placed either in the missile-control system or in the tracking radar. The error performance of these systems for different noise magnitudes is presented. Other considerations such as servo energy requirements are briefly discussed.

INTRODUCTION

In the design of a missile-guidance system certain standard criteria, such as the fastest response, are not always the most useful. This is particularly true in the case of guidance systems which are forced to operate in the presence of certain random unwanted disturbances known as noise. The reason for this is that the effects of the noise can seriously reduce the probability that the missile will hit the target. Furthermore, the sources of noise, being dependent on the target characteristics, cannot be eliminated. Consequently the guidance system should be designed to minimize the miss distance even when the noise is present. This problem will be considered here.

There are two possible design approaches. In the first, the form of the system (i.e., the transfer function) is assumed and an attempt is

made to adjust the existing parameters so as to reduce the effects of
noise (ref. 1). This procedure is not only difficult to apply, but the
ultimate performance is limited by the assumed form. In the second
approach, the form as well as the parameters of the transfer function
are determined so that the noise has the least possible effect on the
performance of the missile; an attempt is then made to devise a system
which has a transfer function approximating this optimum transfer func-
tion. The latter approach will be adopted here.

The problem of determining optimum transfer functions has been
encountered previously in the communication field, and as a result of
this encounter a statistical theory known as Wiener filter theory has
been developed. By the use of this theory it is possible to determine a
unique optimum transfer function which will result in a theoretical lower
limit of mean-square error between the desired and the actual missile
position. Very little work has been done in the application of this
theory to beam-rider guidance. Previous works, references 2 and 3, have
been confined to simple homing systems. The purpose of the present paper,
however, will be to investigate the applicability of this theory to a
beam-rider guidance system.

In the application of this theory to missile guidance it is necessary
to make certain assumptions. Foremost of these is the assumption that the
target and missile move in the same plane, taken in this report to be
horizontal. Other assumptions, such as those relating to the class of
target maneuvers and noise, are discussed in the text. Within these
restrictions, however, the theory may be used to obtain a measure both
of the error performance that might be expected and the difficulties to
be overcome in order to realize this performance.

SYMBOLS

N noise magnitude or zero frequency spectral density, $ft^2/radian/sec$

T_N time constant of the noise spectrum shaping filter, sec

Y_0 optimum closed-loop transfer function

a acceleration of target maneuver, ft/sec^2

k twice the average switching rate of target acceleration, $1/sec$

y_N apparent target displacement from true target center due to noise,
         ft

y_M missile displacement from a space reference, ft

y_T target displacement from a space reference, ft

Φ_N spectral density of noise displacement y_N, $ft^2/radian/sec$

Φ_T spectral density of target displacement y_T, $ft^2/radian/sec$

ϵ error between target and missile position, $y_T - y_M$, ft

ϵ_N component of error ϵ due to noise, ft

ϵ_T component of error ϵ due to target motion, ft

δ control-surface deflection, radians

ψ angle of yaw, $\sim$ i ins

μ_0 optimum open- . ransfer function

μ open-loop transfer function of system approximation to μ_0

ω angular frequency, radians/sec

GENERAL CONSIDERATIONS

Of the many sources of noise which may exist in a guidance system utilizing radar detection, glint noise is one of the most serious. Glint noise is a term that is used to describe a shift in the apparent center of a target as determined by a tracking radar. It is due basically to the variable reflection characteristics of aircraft targets and arises from the relative movement of the various reflecting surfaces. Since the radar utilizes the reflected signal to determine target position, variations in the reflected signal are interpreted by the radar as a shift in the target center. This type of noise is particularly important since it is due fundamentally to the target characteristics and therefore cannot be eliminated by any known radar improvements. The situation is illustrated in figure 1(a) where the true target position is indicated as y_T, and the glint noise is represented by the displacement y_N.

The present report is restricted, for the sake of simplicity, to a two-dimensional study in which the target and missile move in a horizontal plane.[1] The guidance system is considered to be of the beam-rider type, illustrated in figures 1(a) and 1(b). It should be noted here that displacements are referred to a fixed space reference. The function of the guidance system is to make the missile position y_M coincide as closely

[1]The complete three-dimensional problem would require a more complex analysis than used herein. Possibly either the present theory or Wiener's theory for multiple time series (ref. 4) could be applied to this case.

as possible with the actual target position y_T. This requirement would
be relatively easy to satisfy if the tracking radar could locate the
target precisely. However, because of the radar noise the only informa-
tion available to the guidance system is the apparent target position as
illustrated in figure 1(b). For this reason the task is much more diffi-
cult and the missile position may deviate considerably from the actual
target position. The difference, $y_T - y_M$, is denoted on this figure by
the error ϵ, which should be minimized in some manner.

A suitable criterion for judging system performance depends primarily
on the manner in which the system operates. In the case of the beam-rider
system the missile-to-target range is not normally transmitted to the
missile so that the missile cannot know when the target will be reached.
Hence the error should be minimized for all values of range or, equiva-
lently, of time. A mathematically convenient criterion which does not
involve weighting with respect to missile travel time is the mean-square
time average of the error. This criterion will be used herein.

The design of a system normally depends on the inputs to be encoun-
tered, in this case the target motion and noise. Because of their random
nature it is not convenient to define these quantities explicitly as
functions of time, and statistical descriptions are more suitable. Since
it is generally believed that the target motion and noise are uncorrelated
they will be described independently. What follows is a brief discussion
of these inputs.

Intensive effort has been devoted in recent years to the measurement
of radar glint noise. References 3 and 5 through 8 are typical of such
work. The quantity of most general interest in these measurements is the
displacement of the apparent center of the target from the true center,
or y_N, shown in figure 1(a). This quantity can be defined statistically
by means of (1) the amplitude distribution and (2) the power spectral den-
sity. Although the determination of these quantities is somewhat uncertain,
it is generally found that the amplitude distribution is approximately
Gaussian and that the spectral density can be adequately represented by

$$\Phi_N = \frac{N}{T_N^2 \omega^2 + 1} \tag{1}$$

Spectra obtained from any one individual experiment may deviate somewhat
from this form but it is generally considered that the above character-
istic represents a reasonable average of many different experiments.
Examination of noise spectra indicates that the break point $(1/2\pi T_N)$ is
on the order of several cycles per second and for this report will be
taken to be 6 cps corresponding to $T_N = 0.0265$ second. The magnitude
of the spectrum, N, depends on factors such as target size and target
aspect so that the guidance system is usually forced to operate over a
wide range of magnitudes. This range may extend from 7 ft^2/radian/sec
for small targets up to around 30 ft^2/radian/sec for large bombers.

Because of the variation in magnitude due to uncertainty in target size and aspect as well as uncertainty in the noise measurements, this factor becomes of real and practical importance and will be considered herein.

In considering target maneuvers it is difficult to say exactly how a target will maneuver when under attack. However, a reasonable situation might be one in which the target is merely aware of the attack and therefore maneuvers in some severe manner to avoid being hit. Here it will be assumed that the target maneuvers laterally with maximum acceleration alternately in opposite directions. The duration of each acceleration will be a random function determined by some distribution. A reasonable distribution which leads to an easily handled spectral density is the Poisson distribution $[(1/\bar{T})\exp(-T/\bar{T})]$ where T represents the time and $\bar{T}$ the average time between switching of the acceleration. As shown in reference 9 the spectral density of the target acceleration is then described by

$$\Phi_{\ddot{y}_T} = \frac{ka^2}{\pi(\omega^2 + k^2)}$$

Here the quantity a represents the magnitude of the target acceleration normal to the beam, and k is twice the average switching rate, $k = 2/\bar{T}$. The spectral density of the target displacement is then given by

$$\Phi_T \equiv \Phi_{y_T} = \frac{ka^2}{\pi\omega^4(\omega^2 + k^2)} \tag{2}$$

It would appear that there is a problem here concerning the existence of this spectral density because of the ω^4 in the denominator. However, it can be shown that it is possible to use this representation for purposes of computations (see Appendix A). For the tail-chase maneuver to be used in a later example the target acceleration is specified to be ± 1 g at an average period of 5 seconds, which gives $a = 32.2$ ft/sec^2 and $k = 0.4$ switch/sec.

It should be pointed out that a system design based on the target motions described above would operate well against this class of maneuver as a whole. This appears to be a desirable procedure. Nevertheless, without altering the parameters this system would not be expected to operate as well as it could against one particular target maneuver such as a single target turn. Even then, however, it can be shown by simulation studies that systems optimized for the statistical maneuver used herein are essentially optimum for the single turn maneuver as well.

ANALYSIS, RESULTS, AND DISCUSSION

The Wiener Filter Theory

The problem of minimizing the effects of noise can be considered to be one of compromise. At one extreme for which the system response is fast the error becomes large because of the ability of the missile to follow the noise too well. At the other extreme for which the system response is too slow the error becomes excessive because of the difficulty the missile has in following the target maneuvers. The optimum system is one which will compromise these two situations in the best possible manner. More precisely stated the problem becomes: Given the statistical characteristics of the two input quantities, target motion and noise, what is the optimum transfer function which will minimize the mean-square error $\overline{\epsilon^2}$? The answer to this problem can be determined by a statistical theory known as the Wiener filter theory. The final result of this theory is an integral which when evaluated represents the optimum linear transfer function. This transfer function is given by

$$Y_0(i\omega) = \frac{1}{2\pi\Phi^+(\omega)} \int_0^\infty e^{-i\omega t} \int_{-\infty}^\infty \frac{\Phi_T(\alpha)e^{i\alpha t}}{\Phi^-(\alpha)} \, d\alpha \, dt \qquad (3)$$

In this equation Φ^+ and Φ^- have the following meaning: If Φ is the spectral density defined by the equation

$$\Phi = \Phi_T + \Phi_N$$

then Φ^+ and Φ^- must satisfy

$$\Phi^+ \Phi^- = \Phi$$

where Φ^+ has poles and zeroes only in the upper half of the complex ω plane and Φ^- only in the lower. More details are given in Appendix A. The derivation of equation (3) is beyond the scope of this report; for this derivation see references 4 and 10.

The transfer function Y_0 of equation (3) is a mathematical representation of the box in figure 1(b). According to the method of derivation, the transfer function must be physically realizable which means that the system is not required to respond to an input before that input occurs.

It might be pointed out that a general solution exists which involves the problem of prediction as well as filtering. However, since it is apparent from figure 1(a) that we are interested only in the missile

position coinciding with the present target position, the prediction aspect has been eliminated from the general solution in writing equation (3).

Restrictions involved in the theory.- There are certain restrictions implicit in the derivation of equation (3) so that the validity of its application to the beam-rider guidance system depends on how well these restrictions are met. First, the input quantities, target motion and noise, must be stationary random series (see ref. 1 for a detailed definition) and be defined by corresponding power spectra which are continuous. A discontinuity in the spectrum might be due to a predictable component such as a sine wave; such components must be eliminated from the input before the theory can be applied. It is generally believed that displacements at the target are approximately stationary random series (ref. 3). Since the beam-rider system operates from these displacements, the inputs to the beam-rider system are also stationary random series. (In contrast, the inputs to a proportional-navigation guidance system are nonstationary random series since angular inputs are measured by the missile itself and the angles tend to become larger as the range decreases.) Second, the solution is based on linear theory and furthermore is applicable only to a system with constant coefficients in its differential equation. On the other hand, the kinematic loop of guidance systems generally involves a time-variable range factor. In particular for the beam-rider system, the time-variable factor is the ratio of the launcher-to-missile and the launcher-to-target ranges. However, since the miss is determined primarily by what happens near the end of flight, during which the variation in this ratio is small, it is reasonable to assume that the requirement of constant coefficients is approximately met.

Evaluation of the optimum transfer function.- Within the above restrictions, it is possible to use equation (3) to evaluate first, the general form of the optimum transfer function and second, the numerical constants. As an illustration of this method the theory has been applied with certain simplifying assumptions as discussed in Appendix A to the target motion and noise characteristics described earlier. For this case, then, equation (3) can be evaluated to give the general form of the optimum closed-loop and corresponding open-loop transfer functions as follows:

$$Y_0(s) = \frac{(T_\alpha^2 s^2 + 2\zeta_\alpha T_\alpha s + 1)}{(T_\beta s + 1)(T_\gamma^2 s^2 + 2\zeta_\gamma T_\gamma s + 1)} \tag{4}$$

$$\mu_0(s) = k_\mu \frac{(T_\alpha^2 s^2 + 2\zeta_\alpha T_\alpha s + 1)}{s^2(T_\lambda s + 1)} \tag{5}$$

These equations can be evaluated for any specific case and, as an example, for the specific target motion and noise characteristics given below equation (2) and with the magnitude of the noise, N, chosen to be 15 ft^2/radian/sec, the optimum transfer functions become

$$Y_O(s) = \frac{(0.855\ s^2 + 1.41\ s + 1)}{(0.687\ s + 1)(0.490\ s^2 + 0.727\ s + 1)} \tag{6}$$

$$\mu_O(s) = 7.42\ \frac{(0.855\ s^2 + 1.41\ s + 1)}{s^2(2.50\ s + 1)} \tag{7}$$

The chosen value of N represents a mid-value between the expected extremes. The significance of this choice will be discussed later.

The corresponding frequency characteristics are possibly more illustrative to the control designer. The solid curves in figure 2(a) show the optimum closed-loop characteristic for the above conditions and describe the characteristics of the box in figure 1(b). It might be noted that since the break point of the noise spectrum occurs at 6 cps, it is essentially flat over the frequency range of importance of the optimum transfer function. Since actual systems are usually designed on an open-loop basis, the corresponding open-loop characteristic is shown in figure 2(b). The considerations involved in achieving the characteristics of figure 2 are discussed in later sections.

RMS error performance.- Although the Wiener filter theory can be used to define the optimum transfer function, it does not give the minimum error directly. This must be evaluated from the optimum transfer function. With the earlier assumptions as to the form of the target motion and noise, reference 1 shows that the total mean-square error is composed of target and noise components which can be evaluated according to equation (8).

$$\overline{\epsilon^2} = \overline{\epsilon_T^2} + \overline{\epsilon_N^2}$$

$$= \int_{-\infty}^{\infty} |1 - Y_O(i\omega)|^2 \Phi_T(\omega)d\omega + \int_{-\infty}^{\infty} |Y_O(i\omega)|^2 \Phi_N(\omega)d\omega \tag{8}$$

The quantity $Y_O(i\omega)$ is obtained from equation (4) by the substitution s = iω. It is possible to evaluate these integrals mathematically but experience has shown that it is easier and more instructive to use graphical techniques. For the example under discussion, evaluation of equation (8) gives:

$$\sqrt{\overline{\epsilon_N^2}} = 13.7\ \text{feet}$$

$$\sqrt{\overline{\epsilon_T^2}} = 6.7\ \text{feet}$$

$$\sqrt{\overline{\epsilon^2}} = 15.4\ \text{feet}$$

The later figure represents, then, the theoretical lower limit of rms error that can be achieved by a linear constant-coefficient transfer function corresponding to the assumed target and noise characteristics.

For later use, the integrands of equation (8) for the previous example are plotted in figure 3. Physically, these curves can be interpreted as the frequency distribution of the energy in the components of error. It is seen that the error energy is concentrated in the frequency range below 2 cps.

As mentioned earlier, it is important to consider a large range of noise magnitudes for a number of reasons. For example, measurements of glint noise are subject to considerable discrepancy and may cover a wide range of magnitudes. Likewise, the noise magnitude may vary due to different sized targets or different attack aspects. In order to assess these effects the minimum error has been determined as a function of the zero frequency noise magnitude N of equation (1) by a procedure identical to that above. This requires the determination of a different transfer function Y_0 for each value of N, as in Appendix A for $N = 15$ ft^2/radian/sec, and the evaluation of the resultant error by equation (8). The result is shown in figure 4 by curve A. Each point on this curve corresponds to a different optimum transfer function and it is possible to use this transfer function as a guide in designing the guidance system. The value of this curve is that it can be used as a standard with which to compare the performance of any system.

Also indicated in figure 4 is the operating range of noise magnitudes corresponding to the class of targets and aspects previously mentioned. At first it might appear that each noise level would require a change in the filter or system so as to maintain optimum performance. However, as indicated by curve B, if a system is optimized only at the mid-value of the noise range shown here as the design value, the performance of this system for other noise levels will deviate to a negligible extent from the optimum over the range of interest. Thus, to obtain near optimum performance over the range of noise magnitudes likely to be encountered it is necessary to optimize only for the design value of noise. This is a very fortunate fact, and one of obvious practical importance. In a similar manner it can be shown that a change in the break point of the noise spectrum would not greatly affect the minimum error curve shown in figure 4 as long as the noise spectrum is essentially flat over the bandwidth of the optimum transfer function of figure 2.

It is interesting to compare these results with the performance obtained by disregarding noise theory in the design, that is, by designing the system for the fastest possible response. Two examples are shown. Curve C illustrates the performance against noise which is obtained for a system with perfect response characteristics (unity transfer function). Curve D illustrates the performance against noise for a more realistic system given in reference 11 (variable-incidence missile). For this case

the important saturating elements were simulated, and the system was designed for the fastest possible response to a step in the beam of 100 feet. The comparison with the optimum of the curves for both of these example systems shows that the performance is significantly poorer than the optimum performance indicated by theory.

Application of Wiener Theory to the Beam-Rider Guidance System

To achieve the optimum results indicated by the Wiener theory it is necessary to design the guidance system with frequency characteristics approximating the optimum specified by figure 2. The difficulties in accomplishing this as well as the modifications which are required are discussed in the following sections.

Limitations in the application of the Wiener theory.- There are several restrictions on the possible forms of the transfer function which can be achieved in practice. The foremost of these is manifested by a consideration of the accelerations required of the missile. For the optimum system, Y_O, the mean square of the required acceleration $\ddot{y}_M$ is expressed by

$$\overline{\ddot{y}_M^2} = \int_{-\infty}^{\infty} |Y_O(i\omega)|^2 \omega^4 \Phi_T(\omega)d\omega + \int_{-\infty}^{\infty} |Y_O(i\omega)|^2 \omega^4 \Phi_N(\omega)d\omega \qquad (9)$$

It is easy to see that for the form of noise spectrum given by equation (1) and the optimum transfer function of equation (4) the spectral density of the acceleration required by the noise increases with frequency at the high frequencies, giving rise to an infinite called-for acceleration. Obviously finite values of acceleration can be obtained only if the transfer function Y_O falls off at high frequencies like $1/\omega^2$ or some greater power.

To the control designer, this restriction is perhaps more readily interpreted in terms of a control-deflection restriction. Assuming, for the sake of the argument, that the aerodynamic transfer function δ/y_M were everywhere linear (which, of course, it is not, since practically, control deflections cannot increase without limit), the analog of equation (9) would become

$$\overline{\delta^2} = \qquad\qquad \overline{\delta_T^2} \qquad\qquad + \qquad\qquad \overline{\delta_N^2}$$

$$= \int_{-\infty}^{\infty} |Y_O(i\omega)|^2 \left|\frac{\delta}{y_M}(i\omega)\right|^2 \Phi_T(\omega)d\omega + \int_{-\infty}^{\infty} |Y_O(i\omega)|^2 \left|\frac{\delta}{y_M}(i\omega)\right|^2 \Phi_N(\omega)d\omega \qquad (10)$$

It can be seen from equations presented in later sections that δ/y_M approaches a constant multiple of ω^2 at high frequencies. Thus, the conclusion can be drawn from equation (10) that finite values for the control deflection will be obtained for any practical system which may be designed only if the system transfer function differs from the optimum at high frequencies.

At first it would appear that any departure of the system transfer function from the optimum Y_O would result in increased rms error performance. However, an examination of the error spectrum indicates that, since the power contained in frequencies above 2 cps is small, attenuating the transfer function above this frequency will not affect the error appreciably. On the other hand, the control motion spectrum of figure 5 shows that attenuation in the transfer function above 2 cps will greatly reduce the required control motion since the major portion of control motion power is concentrated in the higher frequencies. From these considerations it is obviously desirable to attenuate the transfer function Y_O for all frequencies above the bandwidth of appreciable error spectrum.

The attenuation might be supplied in a variety of ways. However, if it is assumed that an ideal filter could be inserted in the system so as to supply infinite attenuation for all frequencies above 2 cps without affecting the response to frequencies below 2 cps, it is still found that for missiles similar to the one studied here deflections required for this frequency band are larger than are available. Specifically, for the missile used herein the control deflection required is about $15°$ rms; since the control deflection is physically limited to about $14°$ (ref. 11), the called-for control motions are larger than are available. Thus it is impossible for the actual control motion to follow the required program and the optimum missile motions cannot be achieved. Actually the value of $15°$ is quite optimistic since the ideal filter does not exist and cannot even be approached very closely. Consequently, the simultaneous achievement of the optimum error spectrum and called-for control motions which do not exceed the limited values are conflicting requirements. The importance of this conflict and the solution of the problem posed by it are considered in the following sections.

Optimum filtering with limiting.- Because of the above restrictions in the application of the Wiener theory, it is generally necessary to make certain modifications in the practical case. In what follows, the effects of limiting and the consequent modifications required are discussed.

Effects of limiting: There are many limiting-type nonlinearities which may exist in a guidance system. Those of the most importance in the beam-rider system are (a) control motion limiting due to mechanical limitations, (b) rate of control motion limiting due to restrictions on servo capability, and (c) radar receiver voltage limiting due to circuit restrictions. Typical values which are used in these studies are

0.25 radian, 5 radians/sec, and 100 volts, respectively. The effects of these limits can be severe and need to be considered in the filtering problem.

By means of Booton's recent theory (ref. 12) it is possible to evaluate the effects of nonlinearities on the over-all system performance. As applied to the beam-rider system, the theory shows that it is possible to approximate each of the saturable elements by a simple gain which can be suitably chosen so that for increased rms inputs to the saturable element this gain is reduced. Analysis shows that in terms of the over-all system characteristics the effects of this gain reduction are: (1) to reduce the open-loop system gain or (2) to increase the time constants in the open-loop transfer function. In general, both effects result in a reduction in bandwidth of the closed-loop frequency response. In this respect, then, limiting effectively results in additional filtering which, of course, extends further into the low-frequency ranges as the rms inputs increase. Thus the system response can become so slow as to result in a large increase in error due to the missile's inability to follow the target motions. Furthermore, inspection of figure 2(b) shows that the Wiener transfer function is conditionally stable; for large rms inputs the Wiener transfer function may become unstable. That this does occur in examples to be discussed later has been demonstrated during REAC studies. Consequently the large gain reductions and phase lags introduced by limiting cannot be tolerated.

Additional filtering: In general, the energy of the saturating quantities contained in the higher frequencies is much larger than that in the lower frequencies. For this reason, purposely introducing filtering into the system so as to attenuate the response at the higher frequencies can be effective in reducing the effects of limiting. This can be accomplished in either of two ways: by inserting additional networks in the guidance system, or by changing those inherent time constants which exist in any actual system and which are not required for the optimum filtering. Because of the complexity of the beam-rider guidance system there are a great number of places where the additional filtering can be introduced. These possibilities are discussed later.

The selection of the desired frequency characteristics of the added filtering is more difficult than the selection of filtering location. In general, the choice of the desired amount will involve a compromise between two extremes: (1) The first extreme is one in which the added filtering is too severe. As the added filtering is extended into the lower frequencies, limiting effects are reduced but, as previously mentioned, the results of limiting can only be eliminated at the expense of increased error. (2) The other extreme is one in which the added filtering is insufficient to avoid serious limiting effects. This extreme is similar to the previous one inasmuch as the effects of limiting are also equivalent to additional filtering. For serious limiting, this equivalent filtering is too severe and results in increased error and possibly instability.

It is apparent that the effects of limiting and of added filtering by means of networks are closely related, in that both are equivalent to the introduction of additional filtering into the system. Extremes of either type reduce the performance of the system sufficiently that the error increases due to the inability of the missile to follow the target.

Design of the Guidance System

In applying the modifications discussed in the preceding paragraphs to the design of the guidance system it is found convenient to consider two separate stages: The first stage involves the design of the system on a linear basis to approximate the optimum Wiener transfer function, while the second involves supplying the additional filtering terms to minimize the limiting effects. The optimum Wiener transfer function may be designed into the missile-control system, or into the tracking radar, or may be apportioned between the two, with the additional filtering in either place. Thus, there are a number of possibilities. Obviously, for linear systems all approaches could be designed so as to produce identical results. In general, this does not hold for nonlinear systems, that is, systems in which limiting occurs, so that in the practical case it is necessary to consider the effect of the filtering location on performance. The relative merits of placing the optimum Wiener filtering first in the missile-control system and second in the tracking radar are discussed in the next two sections.

Missile-control system designed for optimum Wiener filtering.- For the case of missile-control-system filtering it is necessary that the tracking radar be designed fast enough to follow the target motion and the noise, since the missile-control system performs the optimum filtering operation. The first step in the design of the missile-control system is the synthesis of a linear system which matches the optimum Wiener transfer function. It has been shown that it is only necessary to approximate this transfer function over the frequency range of appreciable error. This can be accomplished in many ways, but to achieve this design by a system of conventional form, the design is most easily made by cut-and-try procedures. In this procedure it is desirable to select the aerodynamics as a starting point since the design of the airframe is relatively inflexible compared to the design of the control system. The characteristics of a typical variable-incidence missile were chosen from reference 11. The transfer functions in the yaw plane for this missile are given by the following equations:

$$\frac{\psi}{\delta} = \frac{T_M s + 1}{T_d s (T_a^2 s^2 + 2\zeta_a T_a s + 1)}$$

$$\frac{y_M}{\delta} = \frac{T_b^2 s^2 + 2\zeta_b T_b s + 1}{T_s^2 s^2 (T_a^2 s^2 + 2\zeta_a T_a s + 1)}$$

Table I summarizes the values of the parameters for this missile for a given operating condition which is held fixed in this report. For this missile it is possible to synthesize a linear system to approximate closely the optimum transfer function. One possible system is illustrated in figure 6 and the corresponding over-all open-loop transfer function is given by

$$\mu = \frac{k_2 k_8 V}{T_d + k_8 k_3} \frac{(T_{10} s + 1)(T_4 s + 1)(T_b^2 s^2 + 2\zeta_b T_b s + 1)}{s^2 (T_{11} s + 1)(T_5 s + 1)(a s^3 + b s^2 + c s + 1)} \qquad (11)$$

The derivation of this transfer function is given in Appendix B. It should be noted that the transfer function of equation (11) differs from that of equation (5); however, by choosing the parameters as given in Appendix B and as tabulated in table II, column ①, the two transfer functions can be closely matched over the frequency range of interest as illustrated by the dotted curves of figures 2 and 3. It might be noted that the only restriction on the parameter T_5 in order to keep the two transfer functions closely matched is that it be small. Thus T_5 may be varied somewhat without much penalty.

The particular design discussed above has been chosen only for optimum noise performance so that other considerations important in an over-all design might dictate certain modifications. These possibilities are discussed later. It is also apparent that since the design of the above system has been based on assumptions of linearity the performance indicated above cannot be achieved in practice due to limiting effects. As indicated earlier these limiting effects may result in instability. Hence modifications are required. These modifications in the two cases of additional filtering, first in the tracking radar and second in the missile-control system, are discussed in the following paragraphs.

Additional filtering in tracking radar: The simpler means of introducing additional filtering into the guidance system in order to reduce limiting is to place it in the tracking radar since the added filtering and optimum Wiener filtering are achieved separately by the tracking radar and missile-control system, respectively. This separation has the advantage of allowing the additional filtering to be altered without affecting the optimum filter design of the missile-control system.

The additional filtering in the tracking radar may take innumerable forms. For this study the closed-loop transfer function of the tracking radar was assumed to be of the following form:

$$\frac{y_B}{y_T + y_N} = \frac{\tau_1 s + 1}{\tau_1^2 s^2 + \tau_1 s + 1} \qquad (12)$$

This transfer function was chosen to provide both simplicity and a zero-velocity error system and it can be approximated by properly shaping the networks in the tracking radar. With this system, the amount of filtering can conveniently be varied by a change in the time constant τ_1. At the one extreme for which τ_1 is very small the filtering provided by the tracking radar is negligible, thus resulting in an unstable missile-control system because of limiting effects. As the time constant of the radar is increased, however, more filtering is provided and limiting effects are reduced. By varying τ_1 figure 7 was obtained; it shows the variation of rms error with the natural frequency, $1/2\pi\tau_1$, of the tracking radar. For comparison, the minimum error obtained from Wiener theory is also shown. As would be expected from previous discussion, the optimum operating point occurs at a tracking radar frequency which is within the passband of the missile-control system. At this frequency limiting effects still exist but are not too serious. The increase in error above the Wiener theory result can be attributed to the additional filtering intro-duced in the tracking radar and to the limiting effects in the missile control system. The rms error performance of the system defined by the optimum operating point is given by curve E in figure 8 against a variable noise level.

Additional filtering in missile-control system: An alternative place to introduce additional filtering into the guidance is the missile-control system. This could be introduced by an additional network in the radar receiver. However, since it has been pointed out that the choice of time constant T_5 was somewhat arbitrary, it is possible to increase T_5 to provide this additional filtering. Thus limiting effects can be reduced.

The introduction of added filtering into the missile-control system has the disadvantage that the added filtering also affects the system stability since an increase in a time lag is destabilizing. Hence, added stability is required. This could be provided by altering any of the basic parameters which are responsible for the conditional stability characteristic of the Wiener transfer function. To preserve the low-frequency characteristics of the optimum system as few changes as possible are desired. The parameter T_{11} is a convenient one with which to intro-duce this stability.

The effects of the above changes were investigated by means of analog simulation. Since in the case of missile-control-system filtering the tracking radar is not required to filter, it should be designed to respond quickly. For this purpose the radar transfer function of the form of equation (12) was utilized by choosing the constants so as not appreciably to alter the input spectra. This was accomplished by the choice of a tracking radar natural frequency of 6 cps. The parameters in the missile-control system were altered according to the above discussion. Typical results obtained for the system optimized only at the design value of noise are illustrated by curve F in figure 8. The optimum Wiener perform-ance is repeated here. In general, results similar to curve F can be

obtained by several different combinations of parameters. One of these combinations requiring the fewest modifications from the Wiener system is given by the parameters in column ② of table II. It can be seen that the additional filtering is supplied by an increased radar receiver network time constant T_5 while the added stability is introduced by a decrease in the time constant T_{11}. In this case the increase in error above the optimum is due to the added filtering introduced by the network and limiting, and to the slight alteration in the Wiener system time constant T_{11}.

Tracking radar designed for optimum Wiener filtering.- In this section are considered the results obtainable when all of the optimum filtering is located in the tracking radar. Here the desired optimum tracking radar transfer functions are given by equations (4) and (5), and ideally the missile-control system should have a transfer function of unity. However, for many reasons, principally those arising from nonlinear effects, the latter may be expected to depart considerably from the ideal. Consequently it is desirable to design the missile-control system for the fastest possible transient response within the limitations of these nonlinearities. Two such missile-control systems were considered. The first was a conventional system chosen from reference 11 in which the response time to a step of 100 feet in the beam was minimized. The second was a system optimized for minimum response time to a small enough step so that linearity was not exceeded. The control-system parameters for these systems are given in columns ③ and ④, respectively, in table II.

The rms error performance obtainable for tracking-radar filtering is summarized in figure 8 by curve G. It was found that both the missile-control systems gave essentially the same results. In this case the increase in error above the Wiener optimum is due to slowness of the missile-control system and its failure to follow the beam perfectly because of limiting effects.

It might appear that these limiting effects could be reduced by additional filtering in the tracking radar. This possibility was explored by the addition of a simple first-order filter to the optimum open-loop transfer function given in equation (5). By increasing the time constant of the added filter the tracking-radar performance is deteriorated from the theoretical optimum while the error of the missile-control system is decreased because of a reduction of limiting effects. Results of these combined effects are shown in figure 9 from which it can be seen that the added filtering results in progressively poorer over-all performance. Thus the beneficial effects of decreased limiting are overbalanced by the detrimental effects of altering the tracking-radar transfer function from the optimum.

Comparisons and Other Considerations

The results of various filtering arrangements are compared in figure 8 on the basis of the rms error. If it were not for the nonlinearities the performance for both missile-control system filtering and tracking-radar filtering could be made identical with the performance of the optimum Wiener system. However, it is interesting to find that even in the nonlinear case all arrangements can result in comparable error performance. Some advantage in tracking-radar filtering is apparent from the figure. For comparison, again, the performance of a typical missile-control system which was optimized for fast response in the absence of noise is repeated.

Up to this point only the error performance has been considered. Other factors which are of importance in the over-all evaluation of a guidance system will now be discussed. One such factor is the servo energy required to achieve a given error performance. For a servo system already designed to meet the maximum expected hinge moment, the servo power is proportional to the time average of the sum of the absolute displacements of δ between values at which $\dot{\delta}$ changes sign. Thus the average servo power for a time t is

$$\text{average servo power} \sim \frac{\Sigma|\Delta\delta|}{t}$$

Evaluation of the servo power for the systems discussed above has shown that both methods of missile-control-system filtering require about 22 percent more power than does tracking-radar filtering. Thus in an over-all evaluation based both on rms error performance and servo energy requirements it is apparent that tracking radar filtering is slightly superior.

There may be still other factors of importance in guidance system design even within the framework of the assumptions discussed earlier. Usually these requirements are related to the specific design objectives of the system and may dictate certain modifications such as the choice of filtering location or alterations in certain individual transfer functions of the system. For example, for short-range missiles in which launching errors are not prevented from building up, capture of the beam in minimum time may be important enough that tracking-radar filtering would be preferable to missile-control-system filtering because of the fast response which can be designed into the missile-control system. In other cases it may be desirable to alter the design of the system somewhat for any of a number of reasons. Requirements of simplicity on certain parts of the system or the necessity of using certain fixed and unalterable elements in the design are examples. Another possibility is that for flight conditions in which serious atmospheric turbulence exists it might be necessary to make certain alterations to minimize the response to gusts. An investigation of such factors is beyond the scope of this report.

CONCLUDING REMARKS

This study has considered the application of Wiener filter theory to the optimization of a beam-rider system operating in the presence of noise. The Wiener theory has been used to establish both the theoretical lower limit of error and the desired transfer-function characteristics. Although these transfer-function characteristics might be expected to vary with noise magnitude, it has been found that when the transfer function is chosen to be optimum for only a mid-value of noise the performance for other noise levels deviates to a negligible extent from the optimum.

In general, linear beam-rider systems can be synthesized to produce theoretical optimum performance, but the practical achievement of these results is restricted by limiting of control deflections, rate of control deflections, and radar receiver voltages. With suitable modifications, however, the theory can be useful as a guide in system design. Optimization in the actual nonlinear case is shown to involve two considerations: limiting and additional filtering. It is shown that the important types of limiting tend to result in system instability. However by appropriate placement of additional network filtering it is possible to minimize these limiting effects without serious deterioration of the error performance.

The design of a guidance system is most conveniently accomplished in two stages: The first consists of designing the system on a linear basis to approximate the Wiener transfer function, and the second of supplying the additional filtering terms to minimize the limiting effects. The application of the Wiener theory and the modifications required to arrive at an optimum system design have been illustrated in this report by considering systems in which the optimum Wiener filtering is designed into either the missile-control system or the tracking radar, and additional network filtering supplied in either place. It was found that comparable error performance can be achieved by any of the methods. Consideration of both rms error and servo energy requirements for the cases studied indicates that optimum filtering in the tracking radar is slightly superior to that in the missile-control system.

In modifying the results of the Wiener theory in order to minimize the effects of limiting, the best results were, in general, obtained when the additional filtering was added in such a way as to tend to keep the operation of the system in the linear range. This has suggested the possibility of seeking an optimum solution based on the stipulation that the filtering should restrict the operation of the system to within its linear range. In reference 13 the results of such an analysis are presented and compared to those of the present report.

Ames Aeronautical Laboratory
    National Advisory Committee for Aeronautics
        Moffett Field, Calif., May 11, 1955

APPENDIX A

DETERMINATION OF THE OPTIMUM WIENER FILTER

Wiener's solution for the optimum transfer function Y_O can be expressed by the following equation:

$$Y_O(i\omega) = \frac{1}{2\pi\Phi^+(\omega)}\int_0^\infty e^{-i\omega t}\int_{-\infty}^\infty \frac{\Phi_T(\alpha)e^{i\alpha t}}{\Phi^-(\alpha)}\,d\alpha\,dt \tag{A1}$$

where Φ_T represents the target motion spectral density, and the quantities Φ^+ and Φ^- are defined as the factors of a certain function Φ with poles and zeroes located in the upper and lower half-planes, respectively; the factor Φ is defined as equal to $\Phi_T + \Phi_N$ so that

$$\Phi = \Phi_T + \Phi_N = \Phi^+\Phi^- \tag{A2}$$

For the case to be considered here, the target motion and noise are defined by

$$\Phi_T = \frac{ka^2}{\pi\omega^4(\omega^2 + k^2)} \tag{A3}$$

$$\Phi_N = N \tag{A4}$$

It will be noted that the noise spectrum has been approximated here by a constant in order to reduce the complexity of the calculations. This approximation is valid because the more exact form of the noise spectrum (eq. (1)) is essentially flat over the bandwidth of the optimum transfer function. Use of the more exact form of the noise spectrum affects only the response at the high frequencies which are beyond the range of interest. It should also be pointed out that the use of equation (A3), as such, leads to certain mathematical difficulties in evaluating the right-hand side of equation (A1), because the theory requires that the poles of Φ_T not be located on the real axis. Rigorously, to avoid these difficulties, it is necessary to modify the target motion spectral density to the following:

$$\Phi_T = \frac{ka^2}{\pi(\omega^2 + \eta_1^2)(\omega^2 + \eta_2^2)(\omega^2 + k^2)} \tag{A5}$$

where η_1 and η_2 are any small real numbers. Thus the solution of equation (A1) is a function of $i\omega$, η_1, and η_2. The desired answer is then obtained as the limit of $Y_O(i\omega,\eta_1,\eta_2)$ as η_1 and η_2 approach zero. However, it can be shown that the same answer can be obtained more simply

by taking η_1 and η_2 equal to zero as would be obtained from the rigorous process described above. It will be convenient to retain the η's for a few lines; subsequently they will be dropped. According to equation (A2) then,

$$\Phi(\alpha) = \Phi_T + \Phi_N = N \frac{\alpha^6 + k^2\alpha^4 + (k\alpha^2/\pi N)}{(\alpha + i\eta_1)(\alpha + i\eta_2)(\alpha - i\eta_1)(\alpha - i\eta_2)(\alpha^2 + k^2)} \tag{A6}$$

$$\Phi(\alpha) = N \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha + \alpha_1)(\alpha + \alpha_2)(\alpha + \alpha_3)}{(\alpha + i\eta_1)(\alpha + i\eta_2)(\alpha - i\eta_1)(\alpha - i\eta_2)(\alpha + ik)(\alpha - ik)} \tag{A7}$$

where α_1, α_2, and α_3 represent roots in the upper half-plane. It should be noted that none of these roots are real. From equation (A2) also, it follows that

$$\Phi^+(\alpha) = N \frac{(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)}{(\alpha - i\eta_1)(\alpha - i\eta_2)(\alpha - ik)} \equiv N \frac{p(\alpha)}{(\alpha - i\eta_1)(\alpha - i\eta_2)(\alpha - ik)} \tag{A8}$$

$$\Phi^-(\alpha) = \frac{(\alpha + \alpha_1)(\alpha + \alpha_2)(\alpha + \alpha_3)}{(\alpha + i\eta_1)(\alpha + i\eta_2)(\alpha + ik)} = \frac{-p(-\alpha)}{(\alpha + i\eta_1)(\alpha + i\eta_2)(\alpha + ik)} \tag{A9}$$

According to the definition given above

$$p(\alpha) = (\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)$$

$$\equiv \alpha^3 + b_2\alpha^2 + b_1\alpha + b_0 \tag{A10}$$

and

$$-p(-\alpha) = \alpha^3 - b_2\alpha^2 + b_1\alpha - b_0 \tag{A11}$$

where

$$\left.\begin{aligned} b_2 &= -(\alpha_1 + \alpha_2 + \alpha_3) \\[1mm] b_1 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \\[1mm] b_0 &= -\alpha_1\alpha_2\alpha_3 \end{aligned}\right\} \tag{A12}$$

The inner integral of equation (A1) is

$$I_1 = \int_{-\infty}^{\infty} \frac{\Phi_T(\alpha) e^{i\alpha t}}{\Phi^-(\alpha)} d\alpha = \frac{ka^2}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha t}}{\alpha^2(\alpha - ik)(\alpha + \alpha_1)(\alpha + \alpha_2)(\alpha + \alpha_3)} d\alpha$$

$$= \frac{ka^2}{\pi} \int_{-\infty}^{\infty} f(\alpha) d\alpha \tag{A13}$$

In the above expression it should be noted that it has been assumed η_1 and η_2 are zero in order to simplify the following expressions. Now if α is considered to be a complex variable, the above integration is equivalent to integration over the contour shown below.
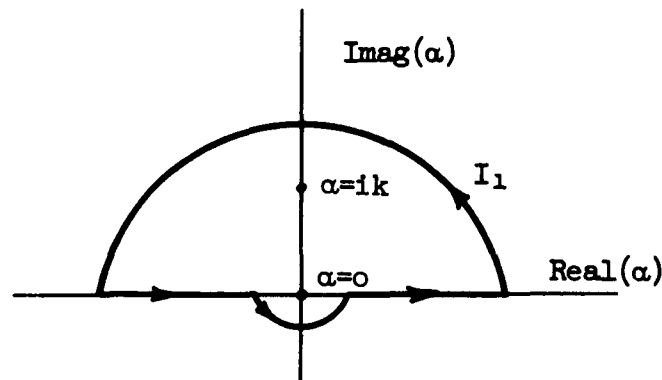
Here the only two poles involved are a second-order pole at the origin (actually at $i\eta_1$ and $i\eta_2$ in the rigorous case) and a first-order pole at $\alpha = ik$ since, as indicated above, $-\alpha_1$, $-\alpha_2$, and $-\alpha_3$ lie in the lower half-plane. The pertinent residues can be found in the usual manner as shown below:

$$\text{Res}(o) = \lim_{\alpha \to o} \frac{d}{d\alpha} [\alpha^2 f(\alpha)]$$

$$= + \frac{t}{kb_0} - \frac{ikb_1 + b_0}{k^2 b_0^2} \tag{A14}$$

$$\text{Res}(ik) = \lim_{\alpha \to ik} [(\alpha - ik) f(\alpha)]$$

$$= \frac{e^{-kt}}{-k^2(ik + \alpha_1)(ik + \alpha_2)(ik + \alpha_3)} = \frac{e^{-kt}}{-k^2[-p(-ik)]} \tag{A15}$$

Hence the integral in equation (A13) becomes

$$I_1 = 2ka^2i\left\{-\frac{ikb_1 + b_0}{k^2b_0^2} + \frac{t}{kb_0} + \frac{e^{-kt}}{-k^2[-p(-ik)]}\right\}$$

$$\equiv 2ka^2i\left[\gamma_1 + \frac{t}{kb_0} + \gamma_2 e^{-kt}\right] \tag{A16}$$

where the definitions of γ_1 and γ_2 are apparent.

The second integration in equation (A1) is merely a Fourier transform and is found as follows:

$$I_2 = \int_0^\infty e^{-i\omega t}I_1 dt$$

$$= 2ka^2i\int_0^\infty\left[\gamma_1 e^{-i\omega t} + \frac{t}{kb_0}e^{-i\omega t} + \gamma_2 e^{-(k + i\omega)t}\right]dt \tag{A17}$$

Thus,

$$I_2 = 2ka^2i\left[\frac{\gamma_1}{i\omega} + \frac{1}{kb_0(i\omega)^2} + \frac{\gamma_2}{k + i\omega}\right]$$

$$= 2ka^2i\ \frac{(\gamma_1 + \gamma_2)(i\omega)^2 + \left(k\gamma_1 + \frac{1}{kb_0}\right)i\omega + \frac{1}{b_0}}{(i\omega)^2(k + i\omega)} \tag{A18}$$

It should be noted that because η_1 and η_2 were assumed to be zero for simplicity, questions concerning the existence of the above integral arise. However if the analysis is made without this assumption, it can be shown that the integral in equation (A17) does exist, and that the limit of this integral as η_1 and η_2 go to zero becomes precisely equation (A18). The coefficients in this equation can be simplified to a more useful form by the following development. From the definitions given earlier in equations (A6) through (A9),

$$\alpha^6 + k^2\alpha^4 + \frac{ka^2}{\pi N} = [p(\alpha)][-p(-\alpha)]$$

which at $\alpha = ik$ becomes

$$\frac{ka^2}{\pi N} = -b_0^2 = [p(ik)][-p(-ik)]$$

When this relation is substituted in the definition of γ_2 given in equation (A16),

$$\gamma_2 = \frac{1}{-k^2[-p(-ik)]} = \frac{p(ik)}{(-k^2)(-b_0{}^2)}$$

$$= \frac{-ik^3 - b_2 k^2 + ib_1 k + b_0}{k^2 b_0{}^2}$$

Thus the coefficients in equation (A18) simplify to

$$\gamma_1 + \gamma_2 = -\frac{ik + b_2}{b_0{}^2}$$

$$k\gamma_1 + \frac{1}{kb_0} = -\frac{ib_1}{b_0{}^2}$$

The transfer function Y_0 can now be found from the preceding equations as

$$Y_0(i\omega) = \frac{I_2}{2\pi\Phi^+} = \frac{-\dfrac{ik + b_2}{b_0}(i\omega)^2 - \dfrac{ib_1}{b_0}(i\omega) + 1}{\left(-\dfrac{\omega}{\alpha_1} + 1\right)\left(-\dfrac{\omega}{\alpha_2} + 1\right)\left(-\dfrac{\omega}{\alpha_3} + 1\right)} \qquad (A19)$$

which, in terms of the conventional complex frequency $s = i\omega$, reduces to the following alternative forms:

$$\left.\begin{aligned}
Y_0(s) &= \frac{\left(-\dfrac{ik + b_2}{b_0}\right)s^2 + \left(-\dfrac{ib_1}{b_0}\right)s + 1}{\left(\dfrac{is}{\alpha_1} + 1\right)\left(\dfrac{is}{\alpha_2} + 1\right)\left(\dfrac{is}{\alpha_3} + 1\right)} \\[2em]
Y_0(s) &= \frac{\left(-\dfrac{ik + b_2}{b_0}\right)s^2 + \left(-\dfrac{ib_1}{b_0}\right)s + 1}{\dfrac{1}{b_0}s^3 - \dfrac{b_2}{b_0}s^2 - \dfrac{ib_1}{b_0}s + 1} \\[2em]
Y_0(s) &= \frac{T_\alpha{}^2 s^2 + 2\zeta_\alpha T_\alpha s + 1}{(T_\beta s + 1)(T_\gamma{}^2 s^2 + 2\zeta_\gamma T_\gamma s + 1)}
\end{aligned}\right\} \qquad (A20)$$

where

$$T_\alpha^2 = -\frac{ik + b_2}{b_0} \qquad\qquad T_\gamma^2 = -\frac{1}{\alpha_2\alpha_3}$$

$$2\zeta_\alpha T_\alpha = -\frac{ib_1}{b_0} \qquad\qquad 2\zeta_\gamma T_\gamma = i\frac{\alpha_2 + \alpha_3}{\alpha_2\alpha_3}$$

$$T_\beta = \frac{i}{\alpha_1}$$

In terms of the equivalent open-loop transfer function, μ_0, of a unity feedback system,

$$\mu_0(s) = \frac{Y_0(s)}{1 - Y_0(s)}$$

$$= \frac{b_0}{ik}\frac{\left(-\dfrac{ik + b_2}{b_0}\right)s^2 + \left(-\dfrac{ib_1}{b_0}\right)s + 1}{s^2\left(\dfrac{s}{k} + 1\right)}$$

$$= k_\mu \frac{T_\alpha^2 s^2 + 2\zeta_\alpha T_\alpha s + 1}{s^2(T_\lambda s + 1)} \qquad\qquad (A21)$$

where

$$k_\mu = \frac{b_0}{ik}$$

The coefficients of the transfer function of the optimum system have been evaluated for the following values of target motion and noise:

$$k = 0.4$$

$$\bar{T} = 5 \text{ sec}$$

$$a = 1 \text{ g}$$

$$N = 15 \text{ ft}^2/\text{radian/sec}$$

Evaluation of the numerator of equation (A6) gives

$$\alpha^6 + k^2\alpha^4 + \frac{ka^2}{\pi N} = \alpha^6 + 0.16\,\alpha^4 + 8.8$$

$$= (\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha + \omega_1)(\alpha + \alpha_2)(\alpha + \alpha_3)$$

where

$$\alpha_1 = i\ 1.456$$

$$\alpha_2 = 1.427\ \exp(i\ 0.543)$$

$$\alpha_3 = -1.427\ \exp(-i\ 0.543)$$

which are all located in the upper half-plane. These roots are then sufficient to determine the optimum transfer function. The constants in equation (A12) give

$$b_2 = -i\ 2.94$$

$$b_1 = -4.201$$

$$b_0 = i\ 2.97$$

Then evaluating the parameters below equation (A20) and in equation (A21) gives the following values:

Parameter	Value
T_α	0.925
ζ_α	.765
T_β	.687
T_γ	.700
ζ_γ	.519
T_λ	2.50
k_μ	7.42

From these parameters the optimum transfer function of equation (A20) becomes

$$Y_0(s) = \frac{(0.855\ s^2 + 1.41\ s + 1)}{(0.687\ s + 1)(0.490\ s^2 + 0.727\ s + 1)} \tag{A22}$$

and from equation (A21) the equivalent open-loop transfer function becomes

$$\mu_0(s) = 7.42\ \frac{(0.855\ s^2 + 1.41\ s + 1)}{s^2(2.50\ s + 1)} \tag{A23}$$

APPENDIX B

MISSILE-CONTROL-SYSTEM APPROXIMATION TO THE OPTIMUM

TRANSFER FUNCTION

An optimum linear design is illustrated in figure 6, and in the following the system equations are derived. From figure 6,

$$\frac{\psi}{E_N}(s) = \frac{k_s(T_M s+1)(T_3 s+1)}{T_d s(T_1 s+1)(T_a^2 s^2+2\zeta_a T_a s+1)(T_3 s+1)+k_s k_3 s(T_M s+1)(T_2 s+1)} \quad (B1)$$

To simplify equation (B1) it is convenient to make $T_3 = T_M$. This choice is not essential but its use leads to more easily handled equations. In certain cases where gust disturbances are serious it may be more desirable to choose T_3 small. With the former assumption, however,

$$\frac{\psi}{E_N}(s)$$

$$= \frac{k_s(T_M s+1)}{s[(T_d T_a^2 T_1)s^3+(T_d T_a^2+T_d 2\zeta_a T_a T_1)s^2+(T_d T_1+T_d 2\zeta_a T_a+k_s k_3 T_2)s+(T_d+k_s k_3)]}$$

$$= \frac{k_s}{T_d+k_s k_3} \frac{T_M s+1}{s(as^3+bs^2+cs+1)} \quad (B2)$$

where

$$\left. \begin{aligned} a &= \frac{T_d T_a^2 T_1}{T_d + k_s k_3} \\[2mm] b &= \frac{T_d(T_a^2 + 2\zeta_a T_a T_1)}{T_d + k_s k_3} \\[2mm] c &= \frac{2\zeta_a T_a T_d + T_1 T_d + k_s k_3 T_2}{T_d + k_s k_3} \end{aligned} \right\} \quad (B3)$$

Thus the entire open-loop transfer function can be written as

$$\mu = \frac{y_M}{\xi} = \frac{k_2 k_s V}{T_d + k_s k_3} \frac{(T_4 s + 1)(T_{10} s + 1)(T_b^2 s^2 + 2\zeta_b T_b s + 1)}{s^2(T_5 s + 1)(T_{11} s + 1)(as^3 + bs^2 + cs + 1)} \quad (B4)$$

It is desirable to choose the parameters in this equation so as to match the optimum transfer function given in equation (A21). Since the system

equation (B4) is of much higher order than the optimum, it is apparent that the matching cannot be accomplished perfectly. However, as shown in the text it is only necessary to approximate the optimum transfer function over a limited frequency band. This can be accomplished by choosing certain terms in (B4) to correspond to terms in (A21). The remaining terms should then be chosen to have negligible effect on the optimum error spectrum.

There are many ways the design can be accomplished. No attempt will be made to investigate all the possibilities. One design, however, is based on the following correspondences:

$$T_4 \longleftrightarrow T_\alpha$$

$$T_{10} \longleftrightarrow T_\alpha$$

$$T_{11} \longleftrightarrow T_\lambda$$

The remaining terms can be chosen in any manner as long as their effect is small over the optimum error spectrum. One of the many possible choices is to factor the cubic to approximate the aerodynamic term in the numerator as follows:

$$as^3 + bs^2 + cs + 1 = (T_b{}^2 s^2 + 2\zeta_b T_b s + 1)(T_L s + 1) \tag{B5}$$

where T_L is arbitrarily chosen to be small. This factoring is not essential to the design but it leads to simple equations for the control-system parameters. For example, for given aerodynamics and choice of T_L the coefficients of the left side of (B5) are determined by

$$\left. \begin{aligned} a &= T_L T_b{}^2 \\ b &= T_L 2\zeta_b T_b + T_b{}^2 \\ c &= T_L + 2\zeta_b T_b \end{aligned} \right\} \tag{B6}$$

Solving equation (B3) then for the control-system parameters gives

$$\left. \begin{aligned} T_1 &= \frac{T_L T_b{}^2 T_a}{2\zeta_b T_b T_L T_a + T_b{}^2 (T_a - 2\zeta_a T_L)} \sim T_L \\ k_s k_3 &= \frac{T_d T_a{}^2 T_1}{T_L T_b{}^2} - T_d \sim T_d \left(\frac{T_a{}^2 - T_b{}^2}{T_b{}^2} \right) \\ T_2 &= \frac{2T_b{}^2}{T_a{}^2 - T_b{}^2} (\zeta_b T_b - \zeta_a T_a) + T_1 + 2\zeta_b T_b \end{aligned} \right\} \tag{B7}$$

Then the value of $k_2 k_8$ is determined from the desired gain:

$$k_2 k_8 = k_\mu \; \frac{T_d + k_8 k_3}{V}$$

It will be noted that since only $k_2 k_8$ and $k_8 k_3$ are specified, one of the three may be chosen arbitrarily. The numerical values of the above parameters have been calculated and are listed in column ① , table II. It should be pointed out that this design is only one of many possible designs. In any particular case certain modifications may be necessary as discussed in the section on comparisons and other considerations.

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TABLE I.- SUMMARY OF AERODYNAMIC PARAMETERS FOR EXAMPLE MISSILE

Parameter	Value
T_a	0.0775
T_b	.0552
T_d	2.087
T_s^2	.000791
T_m	.846
ζ_a	.0536
ζ_b	.0220

TABLE II.- SUMMARY OF CONTROL-SYSTEM PARAMETERS

Parameter	① Linear missile-control system optimum with noise	② Nonlinear missile-control system optimum with noise	③ Nonlinear missile-control system optimum for 100-foot step y_B	④ Linear missile-control system optimum for step y_B
k_2	0.25	0.25	1.0	0.60
T_4	.912	.912	.32	.725
T_5	.01593	.249	.0559	.02
T_{10}	.912	.912	--	--
T_{11}	2.5	80	--	--
k_8	.0462	.0462	.0358	.0358
T_1	.01603	.01603	.025	.025
k_9	44.2	44.2	44.2	55
T_2	.0122	.0122	.0721	.018
T_3	.846	.846	.846	.846
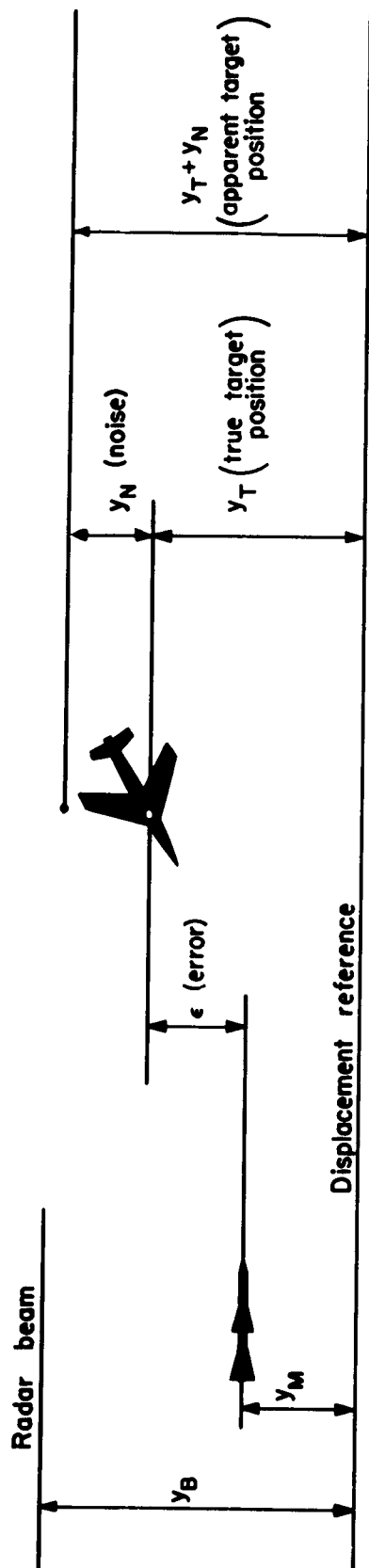
(a) Physical description.
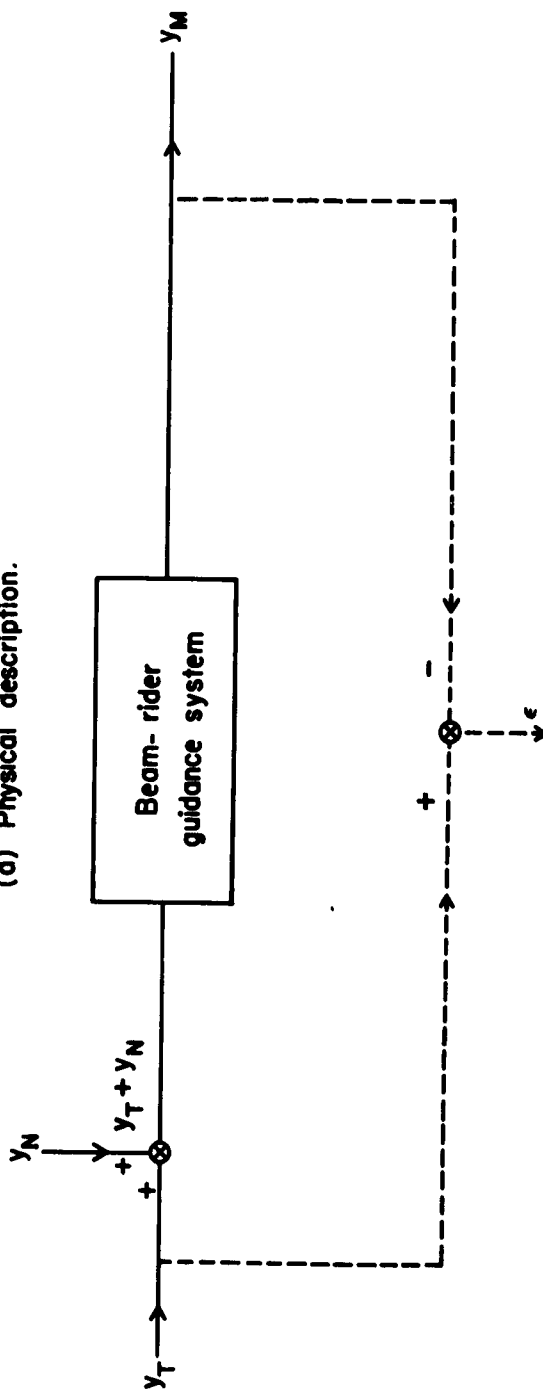
(b) Block diagram.

Figure 1.- Beam-rider guidance system.

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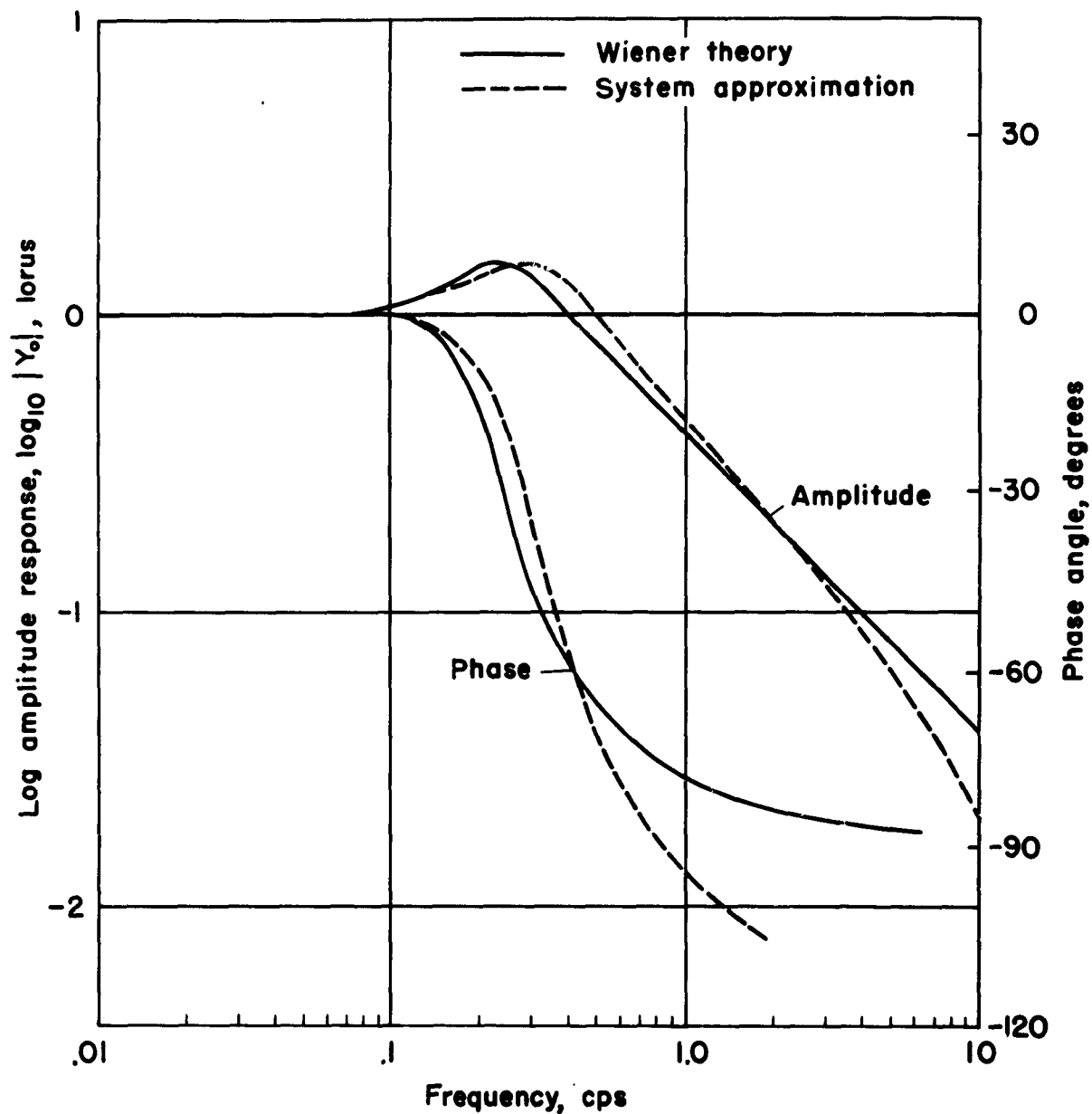
(a) Closed-loop transfer function, Y_o.

Figure 2.- Optimum transfer functions.
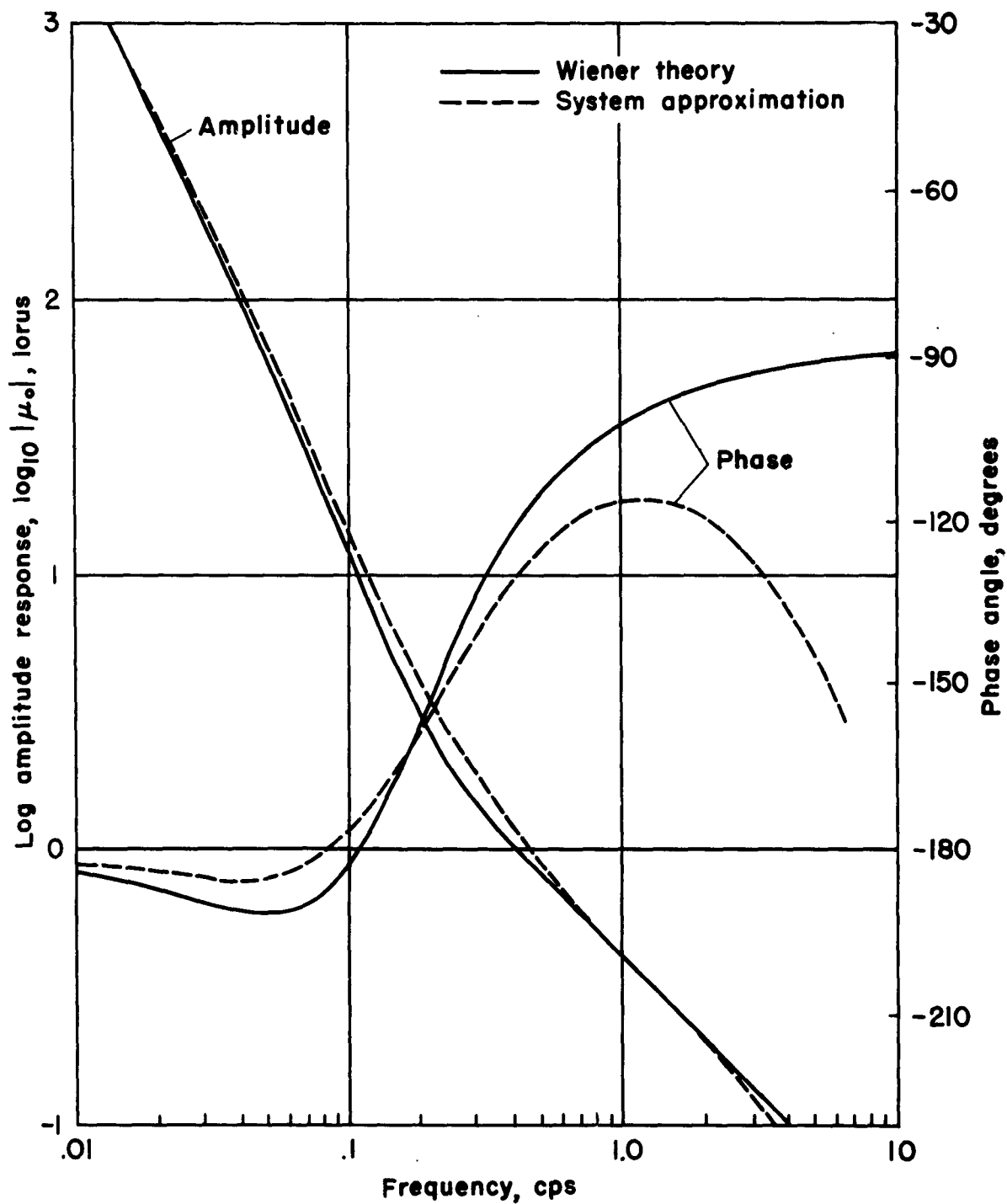
(b) Open-loop transfer function, $\mu_{o'}$

Figure 2.- Concluded.
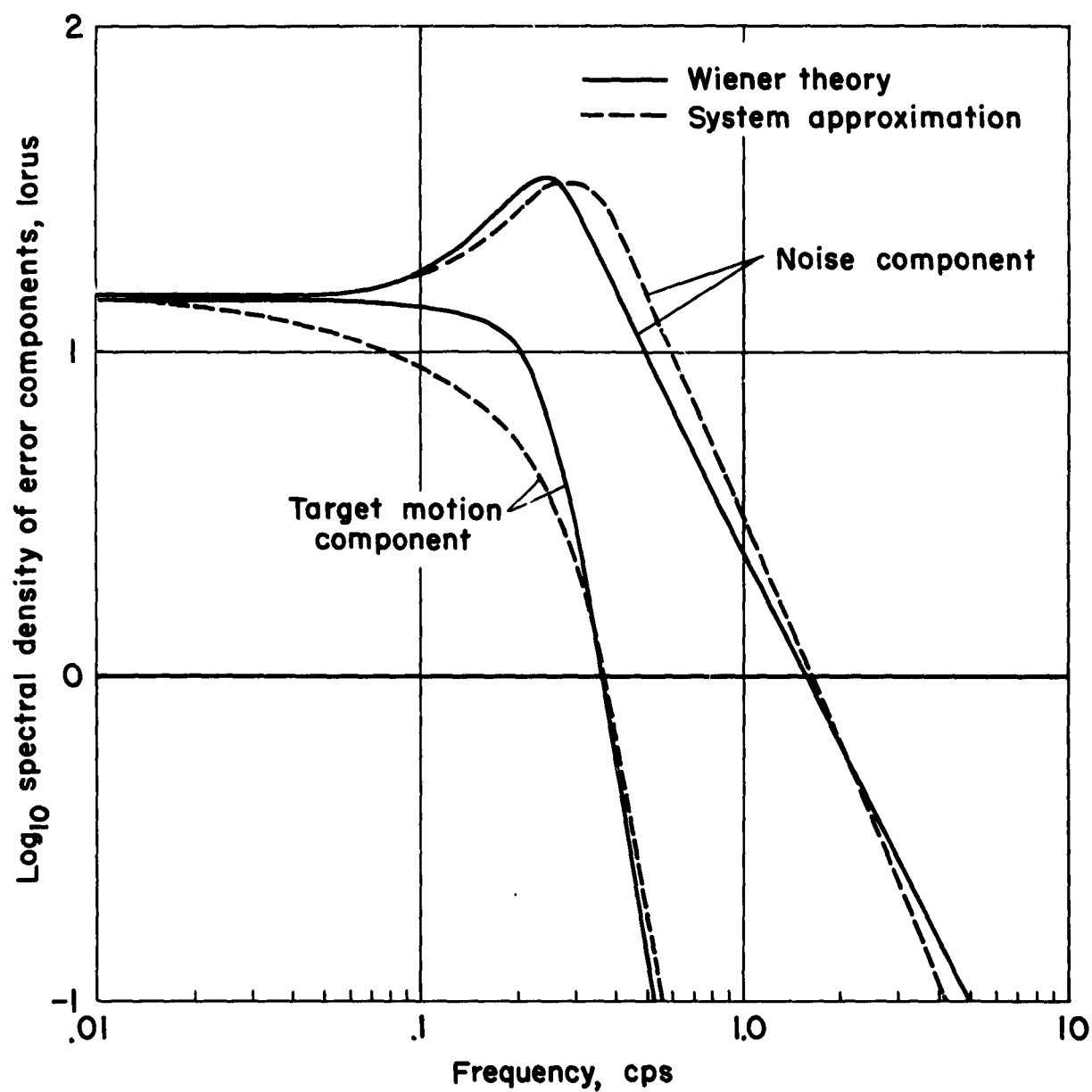
Figure 3.- Spectral density of error components.
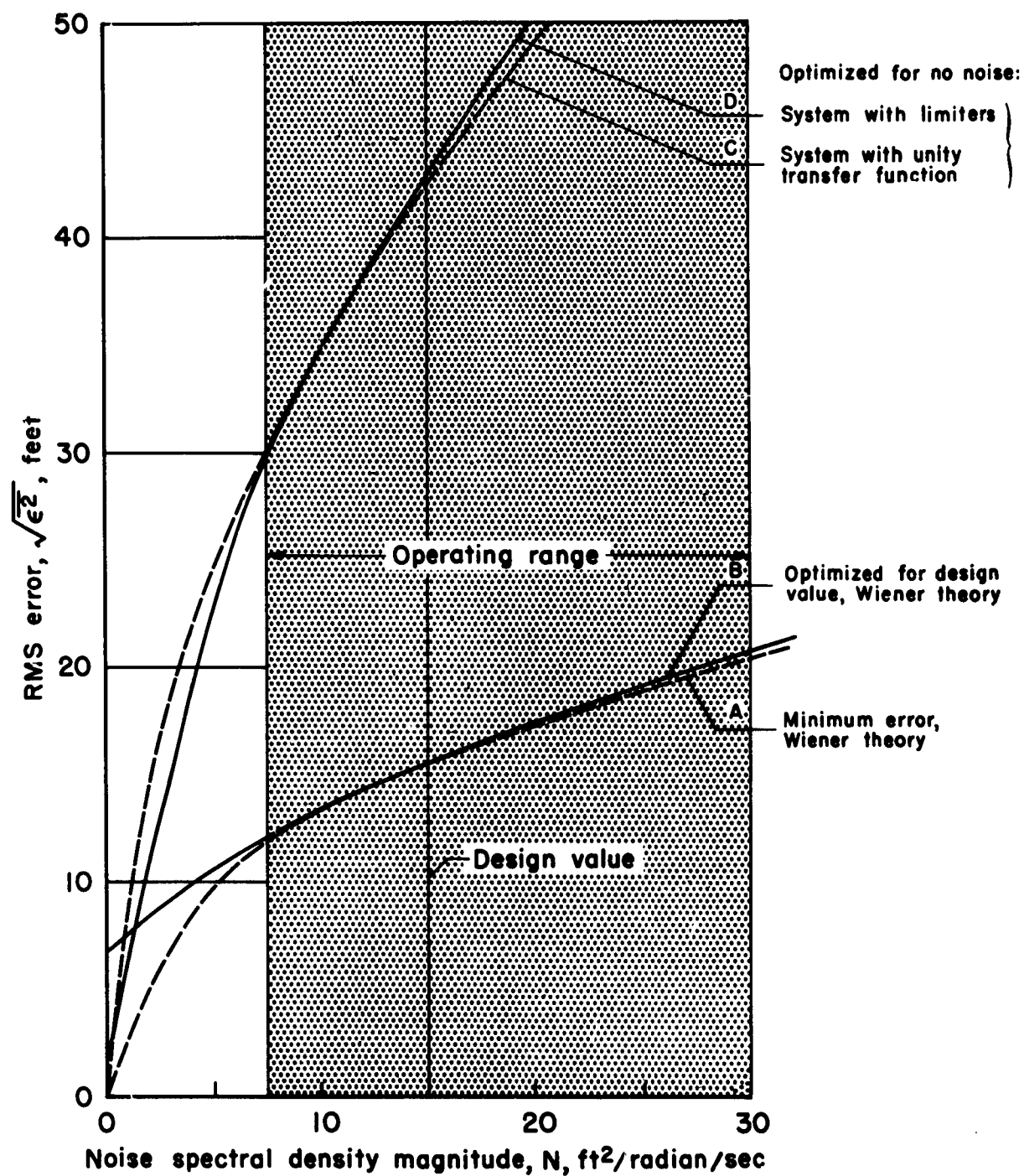
Figure 4.- Optimum performance by Wiener theory.
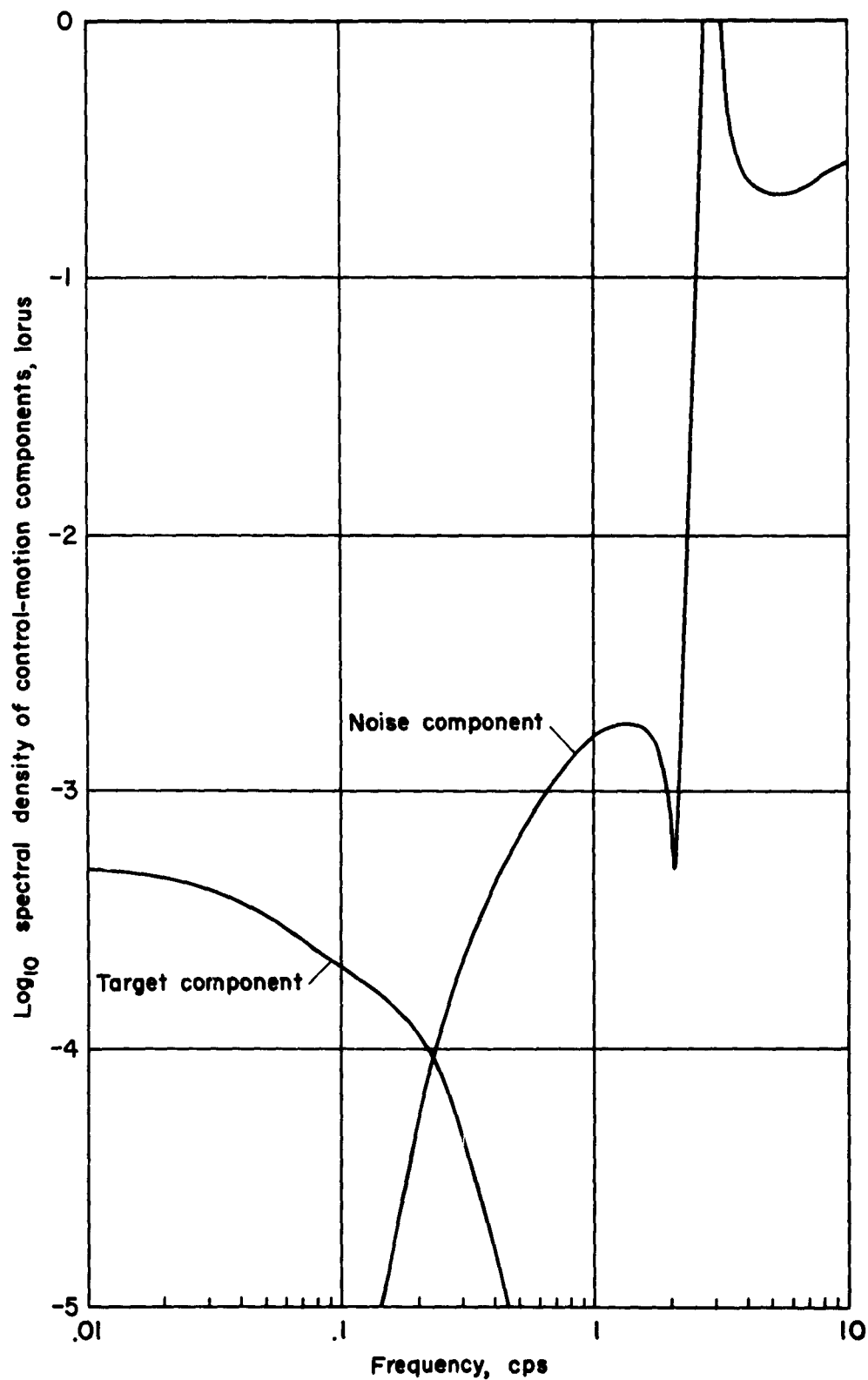
Figure 5.- Spectral density of control motion components.
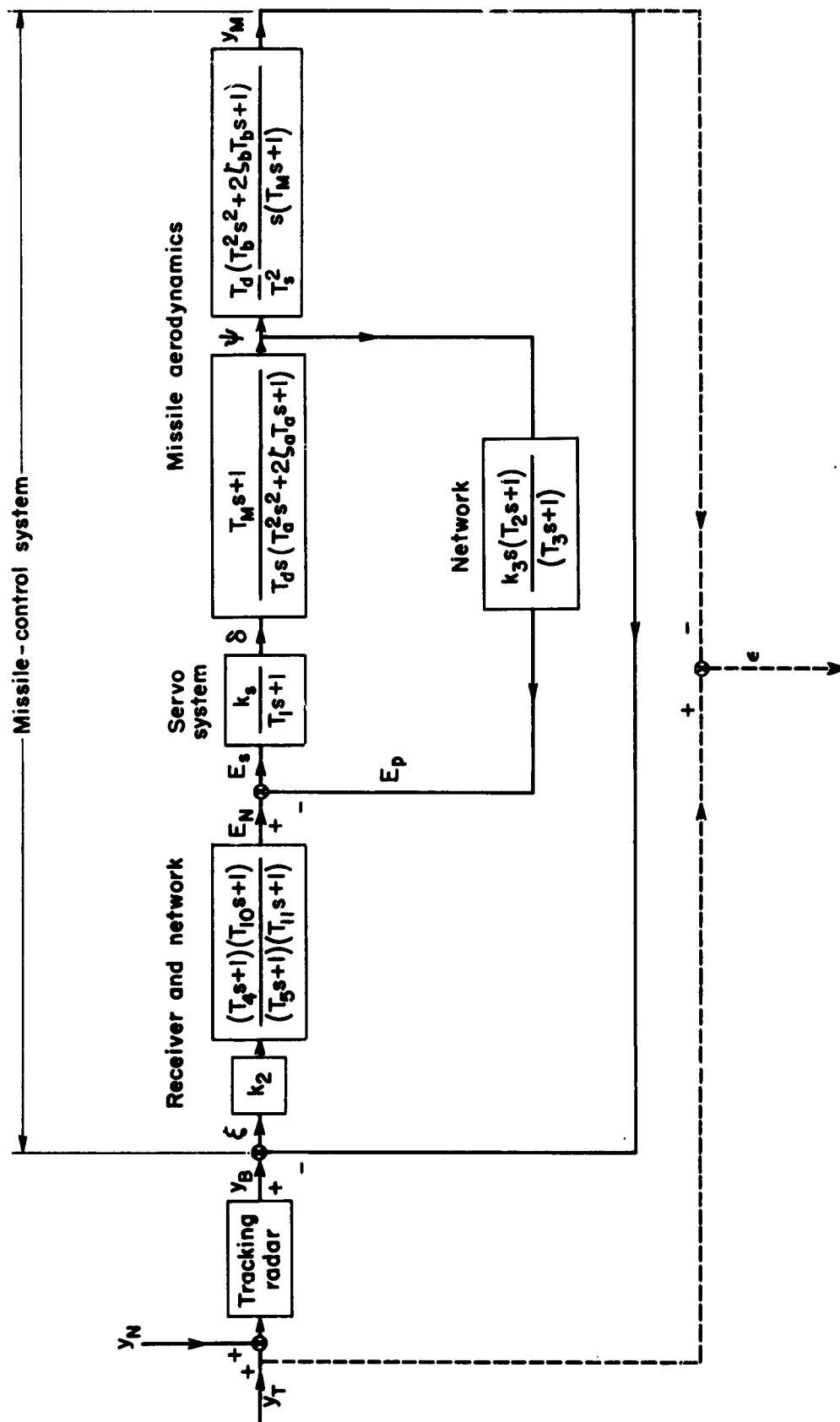
Figure 6.– Linear-system approximation of optimum transfer function.
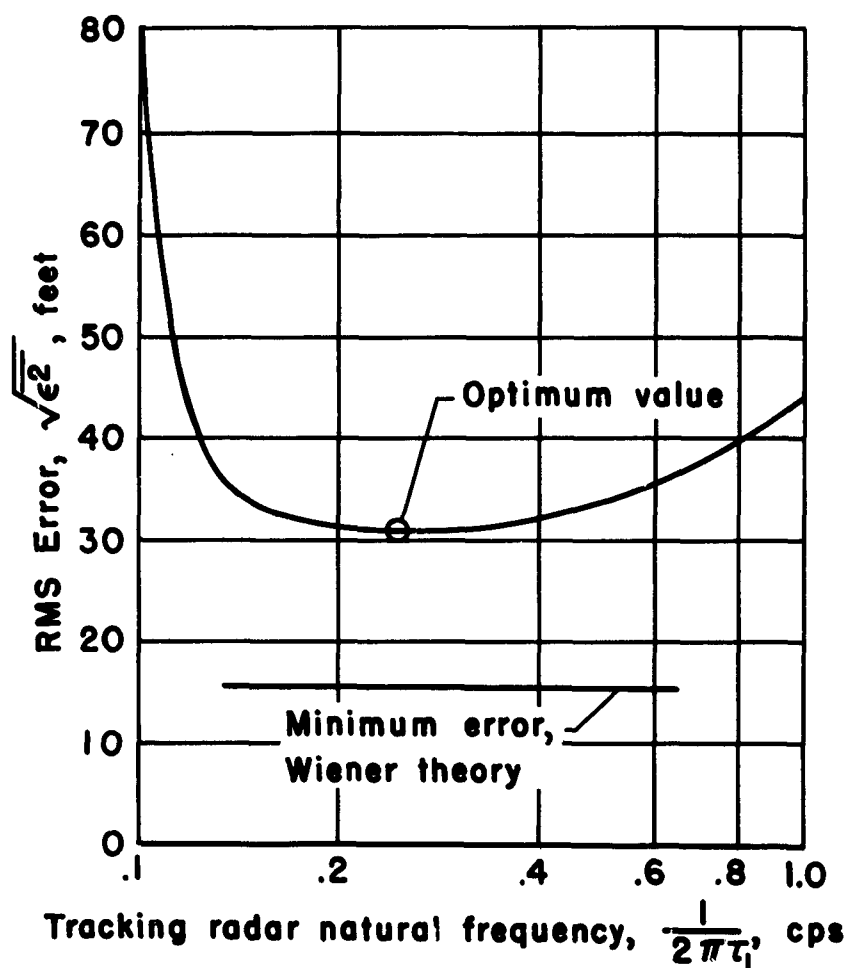
Figure 7.- Effect of additional filtering in tracking radar; missile-control system designed for optimum Wiener filtering.

E. Missile-control system designed for optimum Wiener
   filtering with additional filtering in tracking radar

F. Missile-control system designed for both optimum
   Wiener filtering and additional filtering

G. Tracking radar designed for optimum Wiener filtering
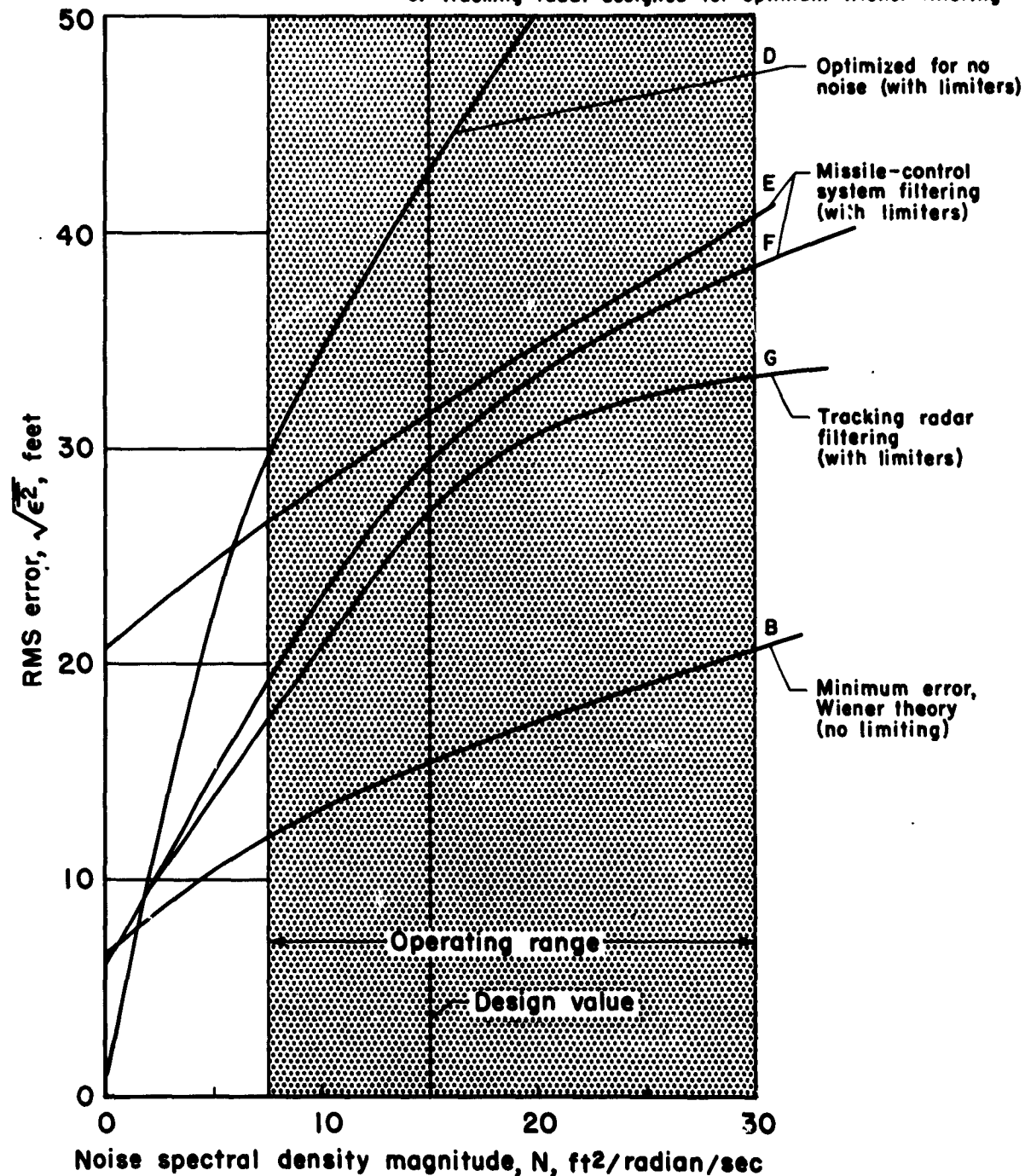
Figure 8.- Effect of practical limits on minimum error; optimum for
design value.
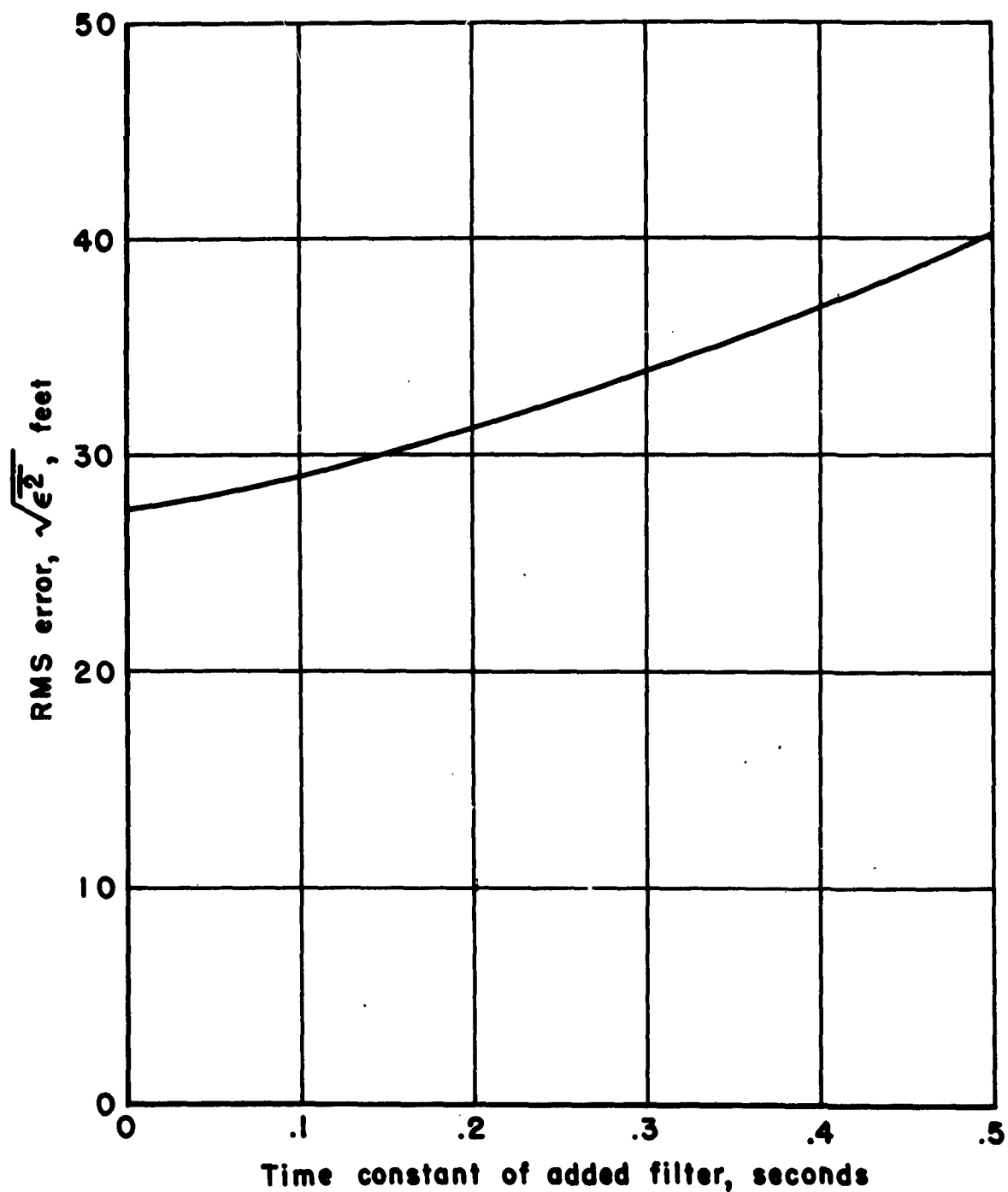
Figure 9.- Effect of additional filtering in tracking radar designed for optimum Wiener filtering.

1. Missiles   (1.7.2)
2. Control, Automatic
            (1.8.2.6)
3. Stabilization, Automatic
            (1.8.8)
I. Stewart, Elwood C.
II. NACA RM A55E11

NACA

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Stewart. August 1955. 40p. diagrs., 2 tabs. (NACA
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